Model Reduction studies in LQG optimal control design for high-speed tilting railway carriages

Argyrios Zolotas, George Halikias, Roger Goodall and Jun Wang

Abstract—The paper studies the utilisation of model reduction techniques, both physical-based and mathematical-based, in designing simplified LQG optimal tilt controllers to improve the curving performance of railway coaches at increased running speed. The schemes make exclusive use of local practical signal measurements, i.e. sensors mounted on the current passenger coach. The fundamental problem related with straightforward classical nulloing-feedback control is presented, while the commercially-used command-driven with precedence scheme is introduced. A combination of simulation results and, a recently proposed, tilt control system assessment method are employed for assessing the performance of the designed LQG controller.

I. INTRODUCTION

A. Tilting trains

Active tilting has become well established in modern railway vehicle technology, with most current high-speed train services in Europe now fitted with tilt and an increasing interest for regional express trains[1]. The concept of tilt is rather straightforward: reduce the lateral acceleration experienced by the passengers, by leaning the bodies of the vehicles inwards on curves, thereby enabling higher vehicle speed operation. These were researched in the 1960s and 1970s, developed for production during the 1980s, and increasingly introduced into service operation during the 1990s.

Early tilt control systems were based solely upon localised-per vehicle measurements (Fig.1a), however it proved impossible at the time to get an appropriate combination of straight track and curve transition performance. Interactions between suspension and controller dynamics (with the sensors being within the control loop) led to control limitations and stability problems. Since then, tilt controllers have evolved in an incremental sense, the end result of which is a control structure which is not optimised from a system point of view. The industrial norm nowadays is to utilise precedence control [1] devised in the early 1980s as part of the Advanced Passenger Train development [2]. In this scheme a bogie-mounted accelerometer from the vehicle in front is used to provide “precedence” (à priori track information) (via appropriate inter-vehicle cable/signalling connections), carefully designed so that the delay introduced by the filter compensates for the preview time corresponding to approximately a vehicle length (Fig.1b).

![Tilt control schemes](image)

Fig. 1. Tilt control schemes

Nevertheless achieving a satisfactory local tilt control strategy remains an important research topic because of the system simplifications and more straightforward failure detection.

B. Model-reduction methods

Model reduction techniques attempt to approximate the dynamic model of the plant by a lower-order system which is easier to control. The problem is especially acute for complex or distributed systems, modelled increasingly with the help of sophisticated finite-element or advanced-dynamics software packages.

Reducing the complexity of the system can offer some clear advantages, e.g related to simplification of the design process and the accompanying simulations, elimination of system modes that are irrelevant to control, identification of crucial characteristics of the system, etc. The main motivation of model reduction, however, comes from the increasing application of modern control-design methodologies (LQG, H∞). These methods are observer-based and thus typically result in controllers of an order comparable to that of the plant (possibly augmented with additional filter/weight dynamics). Since a high-order controller is clearly impractical in most situations, some approximation or model reduction techniques are essential in this case.

There are two main approaches to the model reduction problem. The first, attempts to approximate the input-output characteristics of the plant by a lower order dynamic system, which automatically results in lower-order controllers when modern-control methods are employed. Of course the approximation should be carried out sensibly, so that the
critical modes of the system are not highly affected. For this reason, most model-reduction techniques using this approach are normally accompanied by some form of robust control-design methodology (LQG/LTR, \(\mathcal{H}_\infty/\mu\)) to ensure that the additional uncertainty introduced by the approximation does not have adverse effects of the system’s stability and performance. The second approach to model reduction attempts to apply approximation techniques directly to the (high-order) controller. Controller order reduction [8] may be viewed as a frequency-weighted problem, emphasizing the approximation in critical frequency ranges for the closed-loop system, e.g. near the cross-over frequency. In this work, only the more traditional approach of plant approximation is considered.

An important recent development in the area of model reduction was the emergence of the Hankel-norm approach, which offers tight error bounds on the infinity-norm of the approximation error. In its various forms (balanced-truncation, Hankel-norm optimal, relative-error, frequency-weighted approximation, etc [8], [11]) this approach has resulted in algorithms which can be used effectively for small and medium-size systems (for large-scale problems these methods are not appropriate and Krylov Subspace based methods are preferable [10]).

Despite their success, analytical-based model reduction techniques suffer from the disadvantage that the internal description of the reduced model does not relate in an obvious way to the physical variables which typically describe the original (high-order) system. This is problematic for many engineers who rely on the intuition offered by physical variables to understand and control dynamic systems. In this paper, we attempt to partially bridge the gap between physical and analytical models arising by model-reducing a high-order system, in the context of a case study in the area of tilt-control of railway vehicles. Various reduced-order models of this system, arising both from simplifying physical assumptions and analytical model-reduction techniques are developed, compared and used for optimal control design purposes. The particular objective is to identify simplified optimal control strategies which can be applied to each vehicle independently, i.e. without using precedence control.

From the area of analytic model reduction, although a number of methods exist, we concentrate on those methods which retain certain characteristics (modal frequencies) of the original system, and thus easily relate to engineering intuition. The two methods in this area described later in this paper, involve (i) the system decomposition in slow and fast modes and (ii) its extension presented in [12] via the solution of an optimal Hankel-norm approximation problem with modal constraints.

II. VEHICLE MODELLING

The modelling of the baseline railway vehicle is based upon a linearised end-view model version (Fig. 2), including the lateral and roll dynamics of both the body and the bogie plus: a state from the airspring dynamics, two states from the dynamics of the actuation system and two states from the bogie lateral kinematics (13\textsuperscript{th} order overall).

A pair of linear airsprings represents the vertical suspensions, which only contribute to the roll motion of the vehicle (the vertical degrees of freedom are ignored). The model also contains the stiffness of an anti-roll bar connected between the body and the bogie frame. Detailed wheelset dynamics were not included for simplicity, however the associated effects are incorporated in the model by using an appropriate 2nd order LP filter (bogie lateral kinematics). The filter was characterised by a 5Hz cut-off frequency and 20% damping.

To provide active tilt a rotational displacement actuator, is included in series with the roll stiffness (‘active anti-roll bar (ARB)’ [4]). The actuation system is represented by a position servo in series with the ARB; the parameters chosen such that they gave 3.5Hz bandwidth and 50% damping (closed-loop position servo). The ARB-system is assumed to provide up to a maximum tilt angle of 10 degrees. The advantages of active ARBs results from their relative simplicity, i.e. small weight increase, low cost, easily fitted as an optional extra to build or as a retro-fit.

The mathematical models of increasing complexity, via the Newtonian approach, were developed to encapsulate the lateral and roll dynamics of the tilting vehicle system. The equations of motion are given below with all variables and parameter values provided in Appendix A. For the vehicle body:

\[
\begin{align*}
    m_\text{v} \ddot{y}_\text{v} &= -2k_{\text{sy}}(y_\text{v} - h_1 \theta_\text{v} - y_\text{b} - h_2 \theta_\text{b}) \\
    &\quad - 2c_{\text{sy}}(y_\text{v} - h_1 \theta_\text{v} - y_\text{b} - h_2 \theta_\text{b}) \\
    &\quad - \frac{m_\text{v} v^2}{R} + m_\text{g} \theta_\text{o} - h_1 m_\text{v} \ddot{\theta}_\text{v} \\
    i_{\text{sy}} \ddot{\theta}_\text{v} &= -k_{\text{sy}}(\theta_\text{v} - \theta_\text{b} - \delta_1) \\
    &\quad + 2h_1 \{k_{\text{sy}}(y_\text{v} - h_1 \theta_\text{v} - y_\text{b} - h_2 \theta_\text{b}) \\
    &\quad + c_{\text{sy}}(y_\text{v} - h_1 \theta_\text{v} - y_\text{b} - h_2 \theta_\text{b})\} \\
    &\quad + m_\text{g}(y_\text{v} - y_\text{b}) + 2d_1 \{ - k_{\text{sz}}(d_1 \theta_\text{v} \\
    &\quad - d_1 \theta_\text{b} - k_{\text{sz}}(d_1 \theta_\text{v} - d_1 \theta_\text{b})\} - i_{\text{sy}} \ddot{\theta}_\text{o}
\end{align*}
\]
For the vehicle bogie:

\[ m_b \ddot{y}_b = 2k_{xy}(y_v - h_1 \theta_v - y_b - h_2 \theta_v) + 2c_{xy}(\dot{y}_v - h_1 \dot{\theta}_v - \dot{y}_b - h_2 \dot{\theta}_v) - 2k_{py}(y_b - h_3 \theta_b - y_w) - 2c_{py}(\dot{y}_b - h_3 \dot{\theta}_b - \dot{y}_w) - \frac{m_b v^2}{R} + m_b g \Theta_\theta - h_g m_b \ddot{\theta}_b \]  

(3)

\[ i_{by} \ddot{\theta}_b = k_{s \theta}(\theta_b - \Theta_\theta - \delta_\theta) + 2h_1 (k_{xy}(y_v - h_1 \theta_v - y_b - h_2 \theta_v) + c_{xy}(\dot{y}_v - h_1 \dot{\theta}_v - \dot{y}_b - h_2 \dot{\theta}_v) - 2d_1 (k_{sz}(d_1 \theta_v - d_1 \theta_b) - k_{sz}(d_1 \theta_v - d_1 \theta_b)) + c_{py}(\dot{y}_b - h_3 \dot{\theta}_b - \dot{y}_w) - i_{by} \dot{\theta}_b \]  

(4)

for the (additional) airspring state:

\[ \dot{\theta}_i = - \left( \frac{k_{sz} + k_{cz}}{c_{cz}} \right) \delta_\theta + \frac{k_{sz}}{c_{sz}} \dot{\theta}_v + \frac{k_{cz}}{c_{cz}} \theta_v + \dot{\theta}_b \]  

(5)

for the ARB actuation system:

\[ \delta_a = -22 \delta_a - 483.6 \delta_a + 483.6 \delta_i \]  

(6)

and for the and bogie kinematics:

\[ \ddot{y}_w = -12.57 \ddot{y}_w - 987 \ddot{y}_w + 987 \ddot{y}_v \]  

(7)

Substantial coupling exists between the lateral and roll motions which result in two sway modes combining both lateral and roll movement, and their centres located at points other than the vehicle centre of gravity. An ‘upper sway’ mode, its node appears above the body c.o.g., with predominantly roll movement; and a ‘lower sway’ mode, its node located below the body c.o.g., characterised predominantly by a lateral motion. The modal analysis of the vehicle is shown in Table 1.

### Table I

13th order ARB vehicle model dynamic modes with system states: \([y_v, \theta_v, y_b, \dot{y}_b, \delta_a, \delta_i, \delta_b, y_w, \dot{y}_w]\)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Damping (%)</th>
<th>Frequency (Hz)</th>
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<tr>
<td>Body lower sway</td>
<td>16.5</td>
<td>0.67</td>
</tr>
<tr>
<td>Body upper sway</td>
<td>27.2</td>
<td>1.50</td>
</tr>
<tr>
<td>Bogie lateral</td>
<td>12.4</td>
<td>26.80</td>
</tr>
<tr>
<td>Bogie roll</td>
<td>20.8</td>
<td>11.10</td>
</tr>
<tr>
<td>Bogie kinematics</td>
<td>20.0</td>
<td>5.00</td>
</tr>
<tr>
<td>Actuation system</td>
<td>30.0</td>
<td>3.50</td>
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<tr>
<td>Airspring mode</td>
<td>100.0</td>
<td>3.70</td>
</tr>
</tbody>
</table>

III. MODEL REDUCTION OF VEHICLE DYNAMICS

A. Physically-based approximation (PBA)

The simplest approach to reduce the size of the vehicle model in Section II is to assume a rigid bogie connection to the rail track, i.e. removal of bogie dynamics and kinematics (thus the track inputs have a direct effect on the secondary suspensions). Also to assume an ideal actuator, thus removing the actuation dynamics. Two, physically-reduced, model cases are considered: (PBA1) keep body states plus airspring state (5th order) and (PBA2) keep body states only (4th order). Note that the structure of the reduced states has been retained from the baseline model, i.e. Table II shows the modes of the reduced models, which appear at around the same frequencies, although with changes in the damping (especially for model ‘PBA2’ where no airspring exists the damping for the upper sway mode is further reduced, as one expects).

Fig. 3 illustrates the singular value plot of a sample SISO TF from actuator angle \((u = \delta_a)\) to the effective cant deficiency for 60% partial tilt \(\theta_{ed}\) (see Fig. 1(a)). It can be seen that both 5th and 4th reduced order models are quite close, their modelling error (with respect to the 13th order model) increases significantly at high frequency. The actuator dynamics could be further included in both reduced models to provide an extra \(-40dB/\text{decade}\) roll-off at high frequencies (of course at the expense of 2 extra states, i.e. \(\delta_a, \delta_i\) if necessary; thus providing better approximation of the original features at those frequencies. Note that all disturbance signals have been set to zero for the reduction procedures in Section III.

B. Analytically-based reduction

1) Slow-fast model decomposition: This approach to model reduction decomposes the system into slow and fast modes, retains the former and eliminates the latter. The method relies on the following procedure[13]:

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Let the original system be given in state-space form as
\[ \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \]  \hspace{1cm} (8)
with \( A \) stable. Introduce an orthogonal state-space transformation \( V \) so that \( V^T AV \) is in real-Schur form. In this form, the structure of the transformed matrix is essentially upper triangular, with real eigenvalues appearing on the main diagonal, while (simple) complex-conjugate eigenvalues correspond to \( 2 \times 2 \) blocks extending above and below the main diagonal. The eigenvalues (assumed stable) are ordered according to their magnitude, so that the slow modes are located in the upper-diagonal block. Thus, if \( m \) represents the number of slow modes that we wish to retain,
\[ \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} A \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} A_s & A_{12} \\ O & A_f \end{pmatrix} \]  \hspace{1cm} (9)
where \( A_s \) and \( A_f \) denotes the two blocks containing the slow and fast modes, respectively, and where \( V_1 \) consists of the first \( m \) columns of \( V \). Note that (simple) complex conjugate eigenvalues are not allowed to be split. Next, let \( X \) be the solution of the Sylvester equation
\[ A_s X - X A_f + A_{12} = 0 \]  \hspace{1cm} (10)
It is well-known that the solution to this equation exists and is unique, provided that \( \lambda_i(A_s) - \lambda_j(A_f) \neq 0 \), for all possible \( i \) and \( j \); note that this is automatically satisfied if the eigenvalues are separated by a positive gap in magnitude as is assumed here. Introducing the additional transformation
\[ \begin{pmatrix} I & -X \\ O & I \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} A \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \begin{pmatrix} I & X \\ O & I \end{pmatrix} = \begin{pmatrix} A_s & O \\ O & A_f \end{pmatrix} \]  \hspace{1cm} (11)
which allows the separation of the slow and fast modes through parallel decomposition. Note that in the state vector in the new coordinate system is related to \( x \) via the transformation
\[ z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} x(t) \]  \hspace{1cm} (12)
where \( z_1(t) \) represents the state-vector of the slow part of the realization. This allows us to retain the physical significance of the new state variables \( z_s \), through their link with the original state-vector \( x \); this link could be further developed for LQR-based controller designs.

A sample singular value plot of the original SISO system from actuator angle \( \mu = \delta_a \) to effective cant deficiency for 60% partial tilt (\( \theta_{ecd} \)) and its approximation for: (i) 2 slow modes (4th order, body upper and lower sway) and (ii) 3 slow modes (6th order, body upper and lower sway plus actuator dynamics) is shown in Fig.4. It can be seen that the approximation within the frequency range \([3.2\,\text{rad/s}, 20\,\text{rad/s}]\) is very good for both reduced models. However the 6th order reduced model has successfully retained the DC gain of the original model as well as the original characteristics up to a frequency of \(100\,\text{rad/s} \) and thus it is a better choice for control designs. Note, that the matching of DC gains could be enforced for the 4th order model by following the procedure listed in Appendix B, with \( A_{11} = A_s, A_{22} = A_f, A_{12} = A_{21} = 0 \) and \( [x_1 \, x_2]^T = [z_1 \, z_2]^T \). In such a case the resultant reduced model will be proper (flat response at high frequency), although an appropriate filter can be added to accommodate for the appropriate roll-off (still at the expense of additional states) if necessary.

2) Modal Hankel-norm approximation: A natural extension of the slow-fast decomposition method outlined above is the method described in [12]. The procedure relates to formulating and solving a constrained optimal Hankel-norm approximation problem with modal constraints. The original system \( G(s) \) (assumed stable) is split additively into two systems \( G_1(s) \) and \( G_2(s) \), which can be taken to represent the slow and fast parts of the overall system. \( G_1(s) \) (the slow part) is approximated by a system of the same degree, \( G_1^*(s) \), which shares its \( A \) and \( C \) matrix with \( G_1(s) \) (in the SISO case \( G_1^*(s) \) is constrained to have the same poles as \( G_1(s) \), although not necessarily the same zeros); \( G_2(s) \) is approximated freely by a reduced-order model \( G_2^*(s) \). The overall problem is to choose \( G_1^*(s) \) and \( G_2^*(s) \) (under the above constraints), so that the Hankel norm of the approximation error \( \| G(s) - G_1^*(s) - G_2^*(s) \|_H \) is minimised. The solution to the problem proceeds via a left-coprime factorisation (with inner denominator) and the solution of a (weighted) optimal Hankel-norm approximation problem; more details can be found in [12].

The method was applied to the full-order model (actuator angle \( \mu = \delta_a \)) to effective cant deficiency for 60% tilt compensation on steady swing \( \theta_{ecd} \), with all disturbance signals set to zero. The slow part of the system (4 states) corresponding to the two body sway modes was retained, while the fast part (9 states) was approximated via systems of varying degree \( k \) where \( k \leq 9 \). A sample Bode plot of the original system and its approximation for \( k = 1, 3, 5 \) is shown in Fig.5.

It can be easily seen from the figure, that the procedure successfully retains the slow modes and approximates the fast modes directly via an appropriate filter. The reduced order models can be now utilised for further control designs, with the one based on \( k = 3 \) leading to a 7th order model,

![Fig. 4. Singular value plot for slow-fast reduction](image-url)
IV. TILT CONTROL REQUIREMENTS AND ASSESSMENT APPROACH

A. Requirements

The performance of tilt control systems on the curve transitions is critical; most importantly the passenger ride comfort provided by the tilting vehicle should not be (significantly) degraded compared to the non-tilting vehicle speeds.

The main objectives of any tilt control system are:

1) to provide an acceptably fast response to changes in track cant and curvature (deterministic track features)

2) not to react significantly to track irregularities (stochastic track features)

However, any tilt control system directly controls the secondary suspension roll angle and not the vehicle lateral acceleration. Hence, there is a fundamental trade-off between the vehicle curve transition response and straight track performance. Moreover, for reasons of human perception, designers utilise partial tilt compensation. In such a case the passenger will still experience a small amount of acceleration on steady curve, in order to minimise motion sickness phenomena.

From a control design point of view the objectives of the tilt system can be translated as: increasing the response of the system at low frequencies (deterministic track features), while reducing the high frequency system response (stochastic track features) and maintaining stability.

B. Assessment

The assessment of tilt controllers on curve transition is based upon a combination of the $P_{CT}$ factors approach to effective cant deficiency $\theta_{ecd}$ output, for LQG control design and assess its effectiveness when incorporated to the original model.

Linear Quadratic Gaussian control is well documented in [7],[9] and defines the following state-space plant model

\[ \dot{x} = Ax + Bu + \Gamma w \]  \hspace{1cm} (13)
\[ y = Cx + v \]  \hspace{1cm} (14)

where $w,v$ are (ideally) white uncorrelated process and measurement noises which excite the system, and are characterised by covariance matrices $W,V$ respectively. The separation principle can be applied to first find the optimal control
\[ u = -K_r x \] which minimises (15)
\[ J = \lim_{T \to \infty} \frac{1}{T} E \left\{ \int_0^T [x^T Q x + u^T R u] d\tau \right\}, \tag{15} \]
where \( K_r = R^{-1} B \). Next find the optimal state estimate \( \hat{x} \) of \( x \) where
\[ \dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x}) \tag{17} \]
to minimise \( E \left\{ [x - \hat{x}]^T [x - \hat{x}] \right\} \). The optimal Kalman gain is given by \( K_f = Y C^T V^{-1} \) and \( Y \) is the positive semi-definite solution of the following ARE
\[ [Y - I] \begin{bmatrix} A^T & -C^T V^{-1} C \\ -\Gamma W T^{-1} & -A \end{bmatrix} [I Y] = 0 \tag{18} \]
Weighting matrices \( Q \) (pos. semidefn.), \( R \) (pos. defn.) for control, and \( W \) (pos. semidefn.), \( V \) (pos. defn.) for estimation, can be tuned to provide the desired result. Note that it is also possible to follow the dual procedure, i.e. solve for the state estimate sub-problem and next for the optimal gain sub-problem.

A. Tilt Control Synthesis

Recall that the reduced order model for control has been derived with all disturbance signals set to zero, and also its states are a linear combination of the original states. As a result it is not as straightforward to base the control design strictly on the original (real) aspects of the problem. Note also that the LQG SISO design is a simple extension of the conventional (classical) nulling tilt problem in the optimal control framework.

The view adopted here is to synthesize the tilt controller via LQG theory with the weighting matrices \( Q,R,W,V \) purely considered as tuning parameters until an appropriate design is obtained. In particular, the structure of the LQG tilt compensator is found by shaping the principal gains of the system. First the Kalman Filter is designed via \( W,V \) to obtain a satisfactory return ratio \(-C(sI - A)^{-1} K_f\) at the plant output, with the LQR synthesized via \( Q,R \) such that the return ratio at the output of the compensated plant converges sufficiently close to \(-C(sI - A)^{-1} K_f\) over the frequency range of interest (Loop Transfer Recovery).

Note that for correct tilt compensation on steady curve the LQG compensator should incorporate integral action (thus providing zero sensitivity at zero frequencies). The augmented system is given as
\[
\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & \Gamma \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{w} \tag{19} \]
\[ y = \begin{bmatrix} C \\ 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + v, \tag{20} \]
with the process noise \( w \) from (13) being the integral \( \xi \) of the virtual process noise input \( \tilde{w} \). The covariance matrix of \( \tilde{w} \). However, the eigenvalues of \( A_w \) need to be placed just to the left of the origin for the LQG/LTR solution to exist (for the implementation stage these should move back to the origin for proper integration).

We start with the simplest possible choices for \( \Gamma,V \) equal to \( B,1 \) respectively (still for the SISO model). \( \Gamma = B \) refers to any (virtual) disturbances on the plant acting via the input, rather than the actual track disturbances from the original model. \( \tilde{W} \) is adjusted accordingly to improve (shape) the output return ratio. Fig. 7 illustrates the return ratio \(-C(sI - A)^{-1} K_f\) for various \( W \).

![Fig. 7. Kalman Filter return ratio for various W](imageURL)

The return ratio of the Kalman Filter for \( \tilde{W} = 10 \) with a crossover of approx 10rad/s, is a good choice for recovery. However a simple calculation of the transmission zeros for the design plant reveals a non-minimum phase zeros at approximately 5.5rad/s (this is characteristic for such a setup in tilting trains [3]), thus making full recovery difficult. For illustration, we follow the usual LTR procedure up to the limit of recovery allowed from the nonminimum phase zero (usually the achievable bandwidth of the system is less than half of the RHP zero frequency [9]).

![Fig. 8. LTR at plant output for increasing q = 0.1, 1, 10, 100, 1000](imageURL)

The design of LQR is based upon choosing \( R = 1 \) and...
$Q = CC' + qI$, where $CC'$ relates to the weighting of the reduced order states from the effective cant deficiency output (this has retained information from the original model) and $qI$ is the additional diagonal weighting superimposed relative to the current reduced-order set of states. Fig. 8 illustrates the amount of recovery at plant output for increasing values of $q$. It is seen that there is no point in recovering after $q = 100$. The actual crossover limit is placed by the nonminimum phase zero.

The synthesized LQG controller realization is given by

$$K_{lqg} = \begin{bmatrix} A - BK_r - K_rC' & K_r \\ -K_r & 0 \end{bmatrix}$$

and is 8th order. This was reduced further to a 6th order controller (based on Balanced truncation) without any significant degradation in performance. The sensitivity of the system together with the principal gains of the LQG controller can be seen in Fig. 9. Incorporating the controller in the original system results to approximately the same sensitivity (illustrating the effectiveness of the reduction procedure) compared to the reduced design plant. The principal gain plot of the LQG controller clearly shows its integral action. The time domain results for the lateral acceleration felt by the passengers can be seen in Fig. 10. It illustrates the ideal acceleration (assuming ideal tilt behaviour), the LQG-based and a classical nulling-PI based sample [3]. The deterministic track input used consisted of a curved section with a radius of 1000m superimposed by a maximum track cant angle of 155mm(6°), with a tilting speed of 209km/h. The performance assessment of the controller (based on section IV-B) is presented in Table III. It is seen that although the LQG-based is a straightforward optimal extension of the classical nulling scheme, the performance is much improved (emphasizing robustness with the additional damping injected).

VI. CONCLUSIONS AND FUTURE WORK

The paper discussed on model reduction issues in designing optimal controllers for tilting railway vehicles. The design is based on practical measurements using local vehicle information, i.e. with no ‘a priori’ information on track profile. The optimal nulling-type SISO LQG controller based on the reduced order model of the tilting vehicle, offers a significant improvement of tilting performance compared to the classical nulling-equivalent although both schemes are based on the same concept (and having the same limitations). Shaping the principal gains of the return ratio of the system with the automatic LQG procedure, avoids manually designing networks of classical compensators. In addition the LQG-scheme provides improved robustness properties, although designed on the reduced-order system. Future work is concentrated on the re-formulation of the scheme with extra sensor information and process noise choices and further controller reduction in closed loop. The paper should be of considerable interest to control engineers who have to provide practical solutions but may be put off by the potential complexity of normal model-based control techniques.

REFERENCES


TABLE III
TILT PERFORMANCE ASSESSMENT @ 58(M/S)

<table>
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<tr>
<th>Parameter</th>
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<th>Classical-PI</th>
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<tr>
<td>Lateral accel.</td>
<td>- (actual vs ideal)</td>
<td>9.53</td>
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<tr>
<td>(steady-state)</td>
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<tr>
<td>Roll velocity</td>
<td>- R.M.S. deviation error</td>
<td>16.7</td>
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<tr>
<td>- peak value</td>
<td></td>
<td>0.033</td>
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<tr>
<td>- peak jerk level</td>
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<tr>
<td>$R_T/P$-factor</td>
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<td>- seated</td>
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STOCHASTIC (STRAIGHT TRACK)

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<tbody>
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<td>Passenger comfort</td>
<td>- R.M.S. passive (equiv.)</td>
</tr>
<tr>
<td>- R.M.S. active</td>
<td>4.1</td>
</tr>
<tr>
<td>- degradation</td>
<td>5.67</td>
</tr>
</tbody>
</table>


APPENDIX A- PARAMETER VALUES AND NOTATION

- $y_v$, $y_y$, $y_w$, $y_o$ Lateral displacement of body, bogie, wheelset, track
- $\theta$, $\theta_b$, $\theta_i$ Roll displacement of body, bogie, airspring reservoir
- $\delta_v$, $\delta_w$, $\delta_i$ Track cant, curve radius
- $v$ Vehicle speed
- $m_v$ Half body mass, 19,000(kg)
- $i_{vr}$ Half body roll inertia, 25,000(kg m²)
- $m_b$ Bogie mass, 2,500(kg)
- $h_{ri}$ Bogie roll inertia, 1,500(kg m²)
- $g$ Gravitational acceleration, 9.81(ms⁻²)

--- Values per bogie side ---
- $k_{rz}$ Airspring area stiffness, 210,000(N/m)
- $k_{az}$ Airspring series stiffness, 620,000(N/m)
- $c_{rz}$ Airspring reservoir stiffness, 244,000(N/m)
- $k_{cz}$ Airspring reservoir damping, 33,000(N/m)
- $c_{cz}$ Secondary lateral stiffness, 260,000(N/m)
- $k_{cy}$ Secondary lateral damping, 33,000(N/m)
- $c_{cy}$ Anti-roll bar stiffness/bogie, 260,000(N/m)
- $k_{pt}$ Primary vertical stiffness, 2,000,000(N/m)
- $c_{pu}$ Primary vertical damping, 20,000(N/m)
- $k_{ps}$ Primary lateral stiffness, 35,000,000(N/m)
- $c_{py}$ Primary lateral damping, 16,000(N/m)
- $d_1$ Airspring semi-spacing, 0.90(m)
- $d_2$ Primary vertical suspension semi-spacing, 1.00(m)
- $h_1$ 2ndary lateral susp. height(bogie cog), 0.9(m)
- $h_2$ 2ndary lateral susp. height(bogie cog), 0.25(m)
- $h_3$ Primary lateral susp. height(bogie cog), 0.09(m)
- $h_{b2}$ Bogie cog height(rail level), 0.37(m)
- $h_{b1}$ Body cog height(rail level), 1.52(m)

APPENDIX B - PRESERVATION OF STEADY-STATE GAINS

Assume the following continuous time model

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

using the classical singular perturbation technique [14] parameterize the original system by, an assumed small, scalar parameter $\rho$ such that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du$$

Then by setting $\rho = 0$ solve for $x_1$ (the set of states to be retained) to get

$$\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u$$

$$y = (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u$$

Note that $A_{22}$ must be invertible for the above procedure to apply. The reduced model will preserve the DC characteristics, however there is normally no guaranteed match at high frequencies.